Outline of Topics

I. Approximate Integration

A. Midpoint Rule

1. $\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n)\right]$

$\Delta x = \frac{b-a}{n}$

2. error $E_M = \int_a^b f(x)\,dx - M_n$ $\qquad \bar{x}_i = \frac{x_i + x_{i-1}}{2}$

3. error bounds $|E_M| \le \frac{K(b-a)^3}{24n^2}$ $\qquad f'' \le K$

B. Trapezoidal Rule

1. $\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$

2. error $E_T = \int_a^b f(x)\,dx - T_n$

3. error bounds $|E_T| \le \frac{K(b-a)^3}{12n^2}$ $\qquad f'' \le K$

C. Simpson's Rule

1. $\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$

2. error $\int_a^b f(x)\,dx - S_n = E_S$

3. error bounds $|E_S| \le \frac{K(b-a)^5}{180n^4}$ $\qquad f^{(4)} \le K$

D. examples

1. Use Trapezoidal, Midpoint, & Simpson's Rule to estimate $\int_0^3 \frac{1}{1+y^5}\,dy$ for $n=6$ (#17)

2. How large should n be to guarantee that the Simpson's rule approximation to $\int_0^1 e^{x^2}\,dx$ is accurate to within .00001? (# 22)

II. Improper Integrals

A. Definition: "infinite discontinuity" or "infinite interval of integration"

B.
$$\left.\begin{aligned} \int_a^\infty f(x)\,dx &= \lim_{t\to\infty} \int_a^t f(x)\,dx \\ \int_{-\infty}^b f(x)\,dx &= \lim_{t\to-\infty} \int_t^b f(x)\,dx \end{aligned}\right\} \text{type 1 (infinite interval)}$$

C.
$$\left.\begin{aligned} \int_a^b f(x)\,dx &= \lim_{t\to b^-} \int_a^t f(x)\,dx \\ \int_a^b f(x)\,dx &= \lim_{t\to a^+} \int_t^b f(x)\,dx \\ \int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \end{aligned}\right\} \text{type 2 (infinite discontinuity)}$$

ex. $\frac{1}{x^2}$ $\sqrt{x-3}$

D. Comparison Theorem

E. examples

1. $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\,dx$ (#21)

2. $\displaystyle\int_0^{1} \frac{3}{x^5}\,dx$ (#27)

3. $\displaystyle\int_0^{\infty} \frac{e^x}{e^{2x}+3}\,dx$ (#24)

III. Arc Length

A. Formula $\displaystyle L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \leftarrow \text{Leibniz notation}$$

$$\downarrow$$

$$L = \int_a^b \sqrt{1+(g'(y))^2}\,dy = \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

B. Arc Length Function

$\displaystyle s(x) = \int_a^x \sqrt{1+(f'(t))^2}\,dt$ a is given as a point

C. Examples

1. Find the length of the curve $x^2 = (y-4)^3$ from $P(1,5)$ to $Q(8,8)$ (#20)
2. Find the arc length function of $y = 2x^{3/2}$ with starting point $P_0(1,2)$ (#33)
3. Find the length of the curve $x = \frac{1}{3}\sqrt{y}\,(y-3)$ for $1 \le y \le 9$ (# 11)

IV. Area of a Surface of Revolution

A. $S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$

$S = \int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

x-axis rotation

B. $S = \int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

y-axis rotation

C. examples

1. $y = x^3 \quad 0 \le x \le 2$
2. $y = \frac{x^3}{6} + \frac{1}{2x} \quad \frac{1}{2} \le x \le 1$
3. $x = 1 + 2y^2 \quad 1 \le y \le 2$

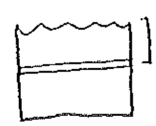
x-axis rotation

4. $y = \sqrt[3]{x} \quad 1 \le y \le 2$
5. $x = \sqrt{a^2 - y^2} \quad 0 \le y \le \frac{a}{2}$

y-axis rotation

V. Applications to Physics & Engineering

A. $F = \rho \cdot g \cdot A \cdot d$



B. $P = \frac{F}{A} = \rho \cdot g \cdot d = \delta d$

C. Moment of the system about the y-axis

$M_y = \sum_{i=1}^{n} m_i x_i$

D. Moment of the system about the x-axis

$M_x = \sum_{i=1}^{n} m_i y_i$

E. Center of mass at $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m} \qquad m = \sum_{i=1}^{n} m_i$$

$$\bar{x} = \frac{1}{A}\int_a^b x\, f(x)\, dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\, dx$$

F. examples

1. Any of #1-10 in §8.3 are great
2. #14, 15, 17
3. # 23, 24, 29 - 33

VI. Sequences

A. $\{a_n\}$ is how we denote a sequence

B. Thm: if $\lim_{x \to \infty} f(x) = L$ and $f(n) = a_n$ then $\lim_{n \to \infty} a_n = L$

C. Limit Laws

D. Squeeze Theorem

E. If $\lim |a_n| = 0$ then $\lim a_n = 0$

F. Thm: if $\lim a_n = L$ and f is continuous at L then $\lim f(a_n) = f(L)$

G. $\{r^n\}$ is convergent if $-1 < r \le 1$

$$\lim r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

H. Monotone Convergence Theorem

1. monotonic
2. bounded

$\Rightarrow$ convergent

VII. Series

A. A series is of the form $\sum_{n=1}^{\infty} \{a_n\}$

B. S_n is the n^{th} partial sum of a series. $S_n = \sum_{i=1}^{n} a_i$

C. A series is convergent if $\lim_{n\to\infty} S_n = S$

D. Geometric series

1. $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$

2. Convergent if $|r|<1$ and $\sum ar^{n-1} = \frac{a}{1-r}$

3. Divergent if $|r| \geq 1$

E. Telescoping sums

F. Harmonic series

G. Thm: If $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim a_n = 0$

H. Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent

I. Laws with series

VIII. The Integral Test

A. The Test:

If $\int_1^{\infty} f(x)dx$ is convergent then $\sum a_n$ is convergent

If $\int_1^{\infty} f(x)dx$ is divergent then $\sum a_n$ is divergent

B. p-series $\sum \frac{1}{n^p}$

1. converges if $p > 1$

2. diverges if $p \leq 1$

C. Remainder Estimates

1. $$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

2. $$S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_{n}^{\infty} f(x)\,dx$$